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GEOMETRY OF VISUAL SPACE: WHAT'S IN A NAME?

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Abstract

The present paper focuses on some conceptual issues pertinent to physiological optics and visual pychophysics. Some notorious problems arising in studies of the geometry of visual space are discussed, and the importance of so-called 'geometric-optical illusions' for these studies is enunciated. Finally, an overview of the topics of the present theme session is given.

> "Nowhere do mathematics, natural sciences, and philosophy permeate one another so intimately as in the problem of space." — Hermann Weyl[1]
>
> "The proper objects of vision constitute an universal language of Nature." — George Berkeley[2]

The space of our everyday life is a rich and complex relational structure, coordinating between no less than three sensory domains: optic, haptic, and kinæsthetic[3]. Visual experience—e. g. contemplating a landscape—gives an account of possible locomotor actions resulting in new haptic and optic data. The experienced unity of space is a kind of synæsthetic experience. To study the connections between different modes of our subjective experience of, and orientation in, space we need to make reference to an 'objective' science of space. This science is *geometry*.

Geometry, as evidenced by its name ($\gamma\varepsilon\omega\mu\varepsilon\tau\rho\iota\alpha$ = earth measuring), developed from a life-world based praxis.[4] However, geometry was the first science to make the transition from sensory experience to rational inference, from empirism to deduction, to a purely mathematical branch of knowledge built upon axiomatic foundations (Euclid), a science of an idealized and in this sense 'objective' space. This notion of space is constituted upon the intersubjective space of haptic and kinæsthetic experience: a space containing physical bodies and also studied or measured by configurations and motions of solid bodies—hence its close relation to mechanics and kinematics. Visual experience plays merely an auxiliary rôle; the drawings of a geometrician are not the proper subject of his study, but only sensory aids, reminders of abstract relations.

Geometry continued the progress toward higher abstraction, via the discovery of non-Euclidean geometries, with a profound change of the discipline's self-understanding. Modern geometry is no longer a science of physical space; the objects of its study are not configurations of physical or ideal bodies, but rather abstract symbolic structures, possible 'geometries'. However, this move does not imply a fatal divorce of the abstract mathematical discipline from the study of sensory experience; it marks rather a new beginning. Indeed, the concept of geometry as the study of *invariants of transformation groups* (F. Klein) parallels the concept of perceptual objects as *invariants of sensory experience* (E. Mach).[5] Interestingly, Klein himself emphasized "the value of space-perception in itself," and pointed out that

> "[t]here is a geometry which is not [...] intended to be merely an illustrative form of more abstract investigations. Its problem is to grasp the full reality of the figures of space, and to interpret [...] the relations holding for them as evident results of the axioms of space-perception."[6]

This note contains *in nuce* a program of a mathematical theory of a *purely visual reality*.

Since Euclid, geometry has been naturally involved in the study of vision, yielding *geometrical optics* as a common theory of vision integrating such disparate domains as the art of pictorial representation (perspective), the construction of optical instruments, and the functioning of the eye. Euclidean geometrical optics seems to provide a rigid scaffolding for visual science,[7] understood as a science of imaging. For the subject matter of physiological optics is *vision*, not the visible; it is a *science of seeing, not of what is seen.*

But is a science of the seen possible at all, and if so, in which form? First steps toward such a science were made by G. Berkeley (*New Theory of Vision*, 1732) and Th. Reid (*Inquiry into the Human Mind*, 1764). Berkeley distinguished between ideas mediated by touch and by sight, the latter being 'signs' or 'marks' of the former, yet *not identical* with them. An intelligence endowed only with the faculty of vision could not arrive at the elementary notions of geometry. While Berkeley was skeptical about applicability of geometry to pure facts of vision, Reid proposed a 'geometry of visibles'[8] as a domain of study on its own standing, and speculated that the visual field must be conceived of as having a spherical geometry.[9]

The conceptual differentiation of 'visual space' (*Sehraum*) from other notions of space was due to the progress of physiology of vision in the 19th century, and intimately related to the problem of *depth vision*—another field of shared interests between sensory physiology and psychology/philosophy of perception. Here we cannot review the rich research agenda of this field: 'local signs', monocular vs. binocular vision, interretinal correlation, determination of the horopter,[10] primary and secondary depth cues,[11] and related aspects of the inverse problem.[12] We only wish to mention some traditional problems and issues relevant for our purpose.

First, consider a frequently raised question: 'Has visual space two or three dimensions?' As usual, under-determined notions lead to ill-posed questions. The manifold of possible visual experience—i. e., distributions of colors and shadings—is of a very high, virtually infinite number of dimensions, while the support of these optical distributions is clearly a two-dimensional[13] and finite domain, which is better named the 'visual field'. It is only matter of definition whether we equate the visual space with the visual field, or with the sub-system of spatial relations in the physical 3-space which can be reconstructed from the actual content of the visual field.[14]

Secondly, we should be aware that the visual space of our experience is not an image plane placed in front of the observer's eye(s), nor can it be identified with a patch of an image surface (e. g. retina). The 'vision *qua* imaging' approach, which models the physical-to-visual space mapping simply as an $\mathbb{R}^3 \to \mathbb{R}^2$ projection, may be useful in some engineering branches of vision science (robotics, 'artificial vision'), but it is inadequate to the primary visual experience.

Thirdly, we come to the problem of the metric structure of visual space. The visual field as such arguably has *no definite metric* prior to visual experience; a problem to which we return later. The question about the metric structure of the visual space arises in the context of optically informed judgments about the external 3-space, or actions in this space. An example of the latter are the well-known experiments with 'visual alleys', on which Luneburg based his theory of a non-Euclidean, hyperbolic geometry of visual space.[15] His claim elicited the interest of experimental researchers, as well as of philosophers[16] studying perception; however, efforts to decide the question experimentally remained remarkably inconclusive. It was found "that no single geometry can adequately describe visual space under all conditions. Instead, the geometry of visual space itself appears to be a function of stimulus conditions."[17]

Attempts to solve principal questions in the laboratory, by an *experimentum crucis*, always show some philosophical naivity. As Poincaré demonstrated for the problem of non-Euclidean character of the physical space,[18] the answer is always a matter of interpretation in convenient theoretical terms; the same conventionalist argument recurs in the study of visual space. The only meaning of the question, 'Is visual space Euclidean or not?', is the following: 'Is there an advantage in using the formalism of non-Euclidean geometry for representation of visual phenomena?'

In view of these controversial results, the question 'Is there a visual space (at all)?'[19] has occasionally been raised; which is, in fact, a confusing pseudo-question. Visual space is nothing just found in nature; it is a concept, a theoretical construct arising from the structure of our sensory experience—exactly as the 3-dimensional physical space is also a construct. The problem is that of the pragmatic *justification* of the concept: do we really need it? I believe that the concept is useful, but that in elaborating a theory of visual space we should not naively transfer notions of geometrical optics applicable only to the physical space. The visual field is not merely a passive projection of the external 3-space.

An evidence for the latter assertion is provided by phenomena known as 'geometric-optical illusions' (GOI),[20] in which perceived lengths, angles or geometric forms are altered by the presence of other elements in the visual field. These phenomena are not random errors or ludicrous deviations from 'veridical perception', as their name may suggest; they are rather stable, robust and impressive manifestations of the laws of visual perception.[21] Quite naturally we encounter GOI phenomena as soon as we attempt to build an intrinsic geometry of the visual field. There are no rigid rods to be moved and apposited; there are only optical facts to be *compared*. Equality or non-equality of lengths, appearance of straight and curved lines, all these visual primitives are dependent, in a holistic manner, on the presence of intersecting or adjacent visual elements.

There have been attempts to interpret some GOIs as the result of mental calculations, based on erroneous interpretations of the presented visual material in terms of the 3rd dimension.[22] These 'explanations', which still can be found in popular science books and some introductory textbooks, are purely speculative and based on improbable premises. Their proponents want us to believe, among other things, that *any* planar drawing (the standard form of stimuli eliciting GOIs) is unconsciously interpreted as an image of a 3-dimensional scene, within which we calculate distances, extents etc. by geometrical reasoning: an untrustworthy and unsupported hypothesis. As said above, vision is not just a projection of the external 3-space to a 2-dimensional image plane. The content of the visual field *may be* a pictorial presentation of a sector of the space 'out there'—or may *not* be. Paraphrasing M. Denis, who reminded us:

> "remember that a picture, before being a war-horse, a nude woman, or any story whatever, is essentially a flat surface covered with colors assembled in a certain order,"[23]

we should remember that the visual field, before being anything else, is a distribution of optical values, colors and their shadings.

The strategy proposed here is exactly the opposite of the 'vision-*qua*-imaging' approach: instead of deriving the geometry of the visual field from the already established geometry of the physical 3-space, we should study visual geometry for its own sake, and attempt to find therein a metric structure, if possible. Only then we will be able to understand how inferences about and actions in the external 3-space are influenced by the intrinsic properties of the visual field.

Some authors have recently observed, in different contexts and obviously independent of each other, a parallelism between the problem of the metric of physical space and that of visual space.[24] In both instances, it is the 'material content' of the space that determines its metric properties. Consequently, talk about a metric *of* the space, and about interactions between material elements *in* the space, are but two different expressions of the same theory: an idea which is now generally accepted in physics, but which may still wait for its full appreciation in the study of visual space.

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This brief overview of 'what's in a name?' of our theme session should help to understand the choice of topics for this session. First, Vincenzo De Risi's paper frames the question of visual space geometry in a historico-philosophical context, with a special focus on Kant. Dejan Todorović examines the 3D-space → 2D-image mappings and their dependence on the observer's position. The next three contributions are concerned with metrical problems in the 2-dimensional visual field, exemplified by GOI phenomena. Aleksander Bulatov reports experimental and modeling work on a modified Müller-Lyer illusion of extent. Werner Ehm presents a mathematical approach to modeling certain types of GOIs, based on a variational principle. Finally, Jiří Wackermann summarizes recent experimental work on the 'filled space expansion' phenomenon (Oppel–Kundt). It is hoped that this selection of papers will contribute to our understanding of visual perception and its intrinsic geometry.

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Notes

[1] Weyl (1949), p. 67.

[2] Berkeley (1732), §147, p. 61f.

[3] Mach (1906), Part I.

[4] Dingler (1911); cf. also Mach (1906), Part II.

[5] Klein (1872) [*Erlangen Program*], esp. p. 463f; Mach (1922), esp. chapter XIV.

[6] Klein (1872), p. 491. Quote from English translation by M.W. Haskell, 'A comparative review of recent researches in geometry', *Bulletin of the New York Mathematical Society*, *2*, 215–249, 1892/93. Note, however, that the term 'space perception' (*Raumwahrnehmung*) does not occur in the original German text; instead, Klein uses the expression *räumliche Anschauung* ≈ 'intuition of space'.

[7] As far as light rays serve as physical realization of perfectly straight lines in Euclidean space, ideal solid bodies can be delineated by pathways of light, that is, by *optical* means. Therefore, somehow paradoxically, the space of geometrical optics is virtually a *haptic* space.

[8] See Section IX (pp. 147–152) of Chapter VI, 'Of seeing' in his *Inquiry*, and also preparatory investigations in Section VIII, where Reid sketches his theory of perception. However, the title 'Of the geometry of visibles' is more programmatic than descriptive; occasional claims making Reid's essay the very beginning of the study of visual space (cf. Wagner, 2006, p. 22) are probably exaggerated.

[9] Reid was credited for having anticipated non-Euclidean geometry (Angell, 1974); this surely deserves qualification. Although the geometry of sphere surface provides a natural model for the elliptic geometry, Reid's talking about the spherical character of the visual field does not make him an antecedent of the 19th century's inventors of non-Euclidean geometries. Cf. also Slowik (2003).

[10] A fervently discussed topic in earlier literature on visual perception. "The significance of the horopter [...] has probably been exaggerated. Not only is its physiological significance obscure, but even its psychophysical definition has become ambiguous." (Richards, 1975)

[11] Banks (2001).

[12] Hatfield (2003).

[13] In the topological sense, that is, regardless of any particular coordinate system, and independent from its existence. For the classical exposition see Poincaré (1913), Chapter III.

[14] If this reconstruction requires a computational scene analysis, based on auxiliary depth cues, or if it

relies immediately upon monocular depth cues is another question which should not be confounded with the dimensionality issue.

[15] Luneburg (1950) and elsewhere; for the experiments on alleys see Hillebrand (1902), Blumenfeld (1913), Indow (1990).

[16] See for example Suppes (1977), or French (1987).

[17] Wagner (1985), p. 493. — Consequently, Wagner proposed a continuum of Riemannian models of variable curvature, with maximum curvature under "totally reduced settings" (i. e., in total absence of distance cues), and null curvature under "information-rich settings", the latter case approaching "the Euclidean ideal of veridical perception." A quarter of century later, the situation seems not to have changed; cf. Wagner (2006), p. 183.

[18] See Poincaré (1929), Chapter IV, 'Experience and geometry'. Cf. also Chapter III, 'Space and geometry', for a discussion of 'pure visual space' vs. 'geometric space', which nicely parallels and complements thoughts of Mach (1906).

[19] See MacLeod and Willen (1995), and Wagner (2006), p. 182f.

[20] Helmholtz (1867), esp. Part 3, p. 562ff; Wundt (1898); Metzger (1975), p. 175ff; Westheimer (2008); Wackermann (2010).

[21] Metzger (1975), esp. p. 185; cf. Mach (1922), p. 8, fn. 1. — The importance of GOIs for the study of vision is evinced by their being used as arguments in the empiricism-vs.-nativism debate; cf. Helmholtz (1867), p. 429f and 804f.

[22] These theories of GOIs still live on Helmholtz' doctrine of 'unconscious inferences' (*unbewusste Schlüsse*) (Helmholtz, 1867, p. 430), a concept infecting parts of psychological literature throughout the 20th century, and re-inforced during last few decades by the rise of the cognitivist-computational paradigm. For a critical evaluation see Hatfield (2002); cf. also Metzger (1975), p. 186f.

[23] "Se rappeler qu'un tableau, avant d'être un cheval de bataille, une femme nue ou une quelconque anecdote, est essentiellement un surface plan recouverte de couleurs en un certain ordre assemblées." (Denis, 1890)

[24] Wagner (2006), p. 183; Westheimer (2008), p. 2141; Wackermann and Kastner (2009), p. 562.

References

Angell, R. B. (1974) The geometry of visibles. *Noûs*, *8*, 87–117.

Banks, E. C. (2001). Ernst Mach and the episode of the monocular depth sensations. *Journal of the History of Behavioral Sciences*, *37*, 327–348.

Berkeley, G. (1732). *An Essay Towards a New Theory of Vision.* In: *George Berkeley, Philosophical Works* (ed. by M. R. Ayers), London: Dent, 1996, pp. 1–70.

Blumenfeld, W. (1913). Untersuchungen über die scheinbare Grösse im Sehraume. *Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, *65*, 241–404.

Denis, M. (1890). Définition du néo-traditionnisme. *Art et Critique*, August 1890. Reprinted in M. Denis, *Théories 1890–1910*, Paris: Rouart & Watelin, 1920, pp. 1–13.

Dingler, H. (1911). *Die Grundlagen der angewandten Geometrie.* Leipzig: Akademische Verlagsgesellschaft.

French, R. (1987). The geometry of visual space. *Noûs*, *21*, 115–133.

Hatfield, G. (2002). Perception as unconscious inference. In: *Perception and the Physical World: Psychological and Philosophical Issues in Perception* (ed. by D. Heyer and R. Mausfeld), New York: Wiley, pp. 115–143.

Hatfield, G. (2003). Representation and constraints: the inverse problem and the structure of visual space. *Acta Psychologica*, *114*, 355–378.

Helmholtz, H. (1867). *Handbuch der physiologischen Optik.* Leipzig: Voss.

Hillebrand, F. (1902). Theorie der scheinbaren Grösse beim binokularen Sehen. *Denkschrift der kaiser-*

lichen Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Classe, *72*, 255–307.

Indow, T. (1990). On geometrical analysis of global structure of visual space. In: *Psychological Explorations of Mental Structures* (ed. by H. G. Geissler), Toronto: Hogrefe & Huber, pp. 172–180.

Klein, F. (1872). Vergleichende Betrachtungen über neuere geometrische Forschungen. Erlangen: Deichert. Reprinted in: *Felix Klein, Gesammelte mathematische Abhandlungen* (ed. by R. Fricke and A. Ostrowski), Vol. I, Berlin, Springer, 1921, pp. 460–497.

Luneburg, R. K. (1950). The metric of visual space. *Journal of the Optical Society of America*, *40*, 627–642.

Mach, E. (1906). *Space and Geometry in the Light of Physiological, Psychological, and Physical Inquiry.* La Salle: Open Court.

Mach, E. (1922). *Die Analyse der Empfindungen und das Verhältnis des Physischen zum Psychischen.* 9th ed., Jena: Fischer.

MacLeod, D. I. A. and Willen, J. D. (1995). Is there a visual space? In: *Geometric Representations of Perceptual Phenomena. Papers in Honor of Tarow Indow on His 70th Birthday.* (ed. by R. D. Luce et al.) Mahwah: Lawrence Erlbaum, pp. 47–60.

Metzger, W. (1975). *Gesetze des Sehens* (3rd ed.). Frankfurt: Kramer.

Poincaré, H. (1913). *Mathematics and Science: Last Essays.* New York: Dover, 1963.

Poincaré, H. (1929). *Science and Hypothesis.* In: H. Poincaré, *The Foundations of Science.* New York: Science Press, pp. 27–197.

Reid, Th. (1764). *An Inquiry into the Human Mind on the Principles of Common Sense.* In: *The Works of Thomas Reid* (ed. by W. Hamilton, 3rd ed.) Edinburgh: MacLachlan and Stewart, 1852, pp. 93–211.

Richards, W. (1975). Visual space perception. In: *Handbook of Perception, Vol. V*, Chapter 10, New York: Academic Press.

Slowik E. (2003). Conventionalism in Reid's 'geometry of visibles'. *Studies in History and Philosophy of Science A*, *34*, 467–489.

Suppes, P. (1977). Is visual space Euclidean? *Synthese*, *35*, 397–421.

Wackermann, J. and Kastner, K. (2009). Paradoxical form of filled/empty optical illusion. *Acta Neurobiologiae Experimentalis*, *69*, 560–563.

Wackermann, J. (2010). Geometric-optical illusions: a pedestrian's view of the phenomenal landscape. In: *Fechner Day 2010* (ed. by A. Bastianelli and G. Vidotto), Padua: International Society for Psychophysics, pp. 171–176.

Wagner, M. (1985). The metric of visual space. *Perception and Psychophysics*, *38*, 483–495.

Wagner, M. (2006). *The Geometries of Visual Space.* Mahwah: Lawrence Erlbaum.

Westheimer, G. (2008). Illusions in the spatial sense of the eye: Geometrical–optical illusions and the neural representation of space. *Vision Research*, *48*, 2128–2142.

Weyl, H. (1949). *Philosophy of Mathematics and Natural Science.* Princeton: University Press.

Wundt, W. (1898). Die geometrisch–optischen Täuschungen. *Abhandlungen der mathematisch-physischen Classe der Königlichen Sächsischen Gesellschaft der Wissenschaften*, *24*, 55–178.