DUAL KLEPSYDRA MODEL OF DURATION DISCRIMINATION

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Abstract

In the present paper the dual klepsydra model is applied to discrimination of temporal durations, assessed by the method of pairwise comparison. Equivalence of the duration comparison and reproduction paradigms is demonstrated; expressions for the theoretical psychometric function in terms of model parameters are given; and an approximative method to estimate the model parameters from an empirical psychometric function is presented. Estimates based on data from two studies on duration discrimination are in good agreement with earlier estimates of relaxation times derived from duration reproduction data. Various aspects of the model (presentation order error, estimates of internal states, cognition of temporal order) are briefly discussed.

Dual klepsydra model of internal time representation [1, 2] was originally designed for the duration reproduction paradigm. The model accounts naturally for qualitative features of data not addressed by other models, and matches experimental data with good accuracy. Here we apply the stochastic version of the model (SDKM) [3, 4] to another experimental paradigm for studies of duration discrimination, namely, to pairwise comparison of temporal intervals.

Duration discrimination in the SDKM

Description of the model

The model consists of two inflow–outflow units (IOU) or, metaphorically, "leaky klepsydræ." Two IOUs are allocated for internal representation of two time intervals, s_1 and s_2, which are presented sequentially to the subject and separated by an inter-stimulus interval w (Fig. 1a). Klepsydra 1 is filled at inflow rate i_1 during the first interval ($0 \le t \le s_1$), and "leaks" thereafter; klepsydra 2 is filled at inflow rate i_2 during the second interval ($s_1 + w \le t \le s_1 + w + s_2$) (Fig. 1b). Afterward, at time $t > s_1 + w + s_2$, the subject has to indicate which of the two intervals was perceived as longer. In terms of the SDKM, the states of both klepsydræ are compared, and the higher one determines the subject's judgment, $J \in \{1, 2\}$. Occasionally we write '$s_1 \gtrdot s_2$' or '$s_1 \lessdot s_2$' for $J = 1$ or $J = 2$, respectively, and read "subjectively longer" or "subjectively shorter."

The dynamics of an IOU is described by a stochastic linear differential equation

$$\mathrm{d}Y_t = (i - \kappa\, Y_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \tag{1}$$

with constant inflow rate i, "leakage coefficient" κ, and a noise term $\sigma\,\mathrm{d}W_t$, where W_t is the standard Wiener process. We will assume that inflow rates i_1, i_2 and dispersions σ_1, σ_2 are identical for both klepsydræ. The model thus has three parameters, κ, i and σ. Klepsydræ states are unobservable, and thus arbitrarily scalable; therefore, only the ratio $\gamma \equiv i/\sigma$ can be determined uniquely, and the model is fully specified by two parameters, κ and γ. The related quantities κ^{-1} and γ^{-2} are referred to as *relaxation time* and *diffusion time*, respectively, of the system.

Solution to eq. (1) under the initial condition $Y_0 = 0$ is

$$Y_t = \frac{i}{\kappa}\left(1 - \mathrm{e}^{-\kappa t}\right) + \sigma\, U_t\,,$$

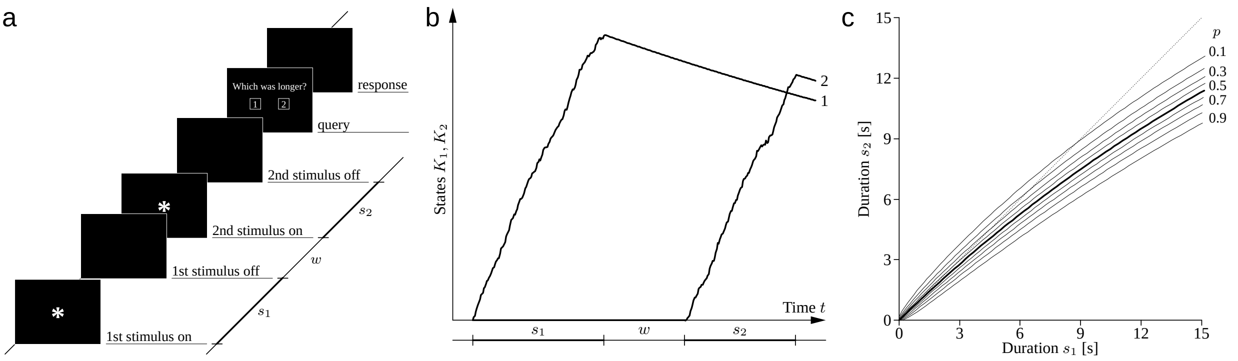

Figure 1. (a) Time chart of an experimental trial. (b) Sample trajectories of states accumulated at klepsydræ 1, 2. (c) Contour plot of a PMF with parameters γ = 3.162 s $^{-1/2}$, κ = 0.02 s^{-1}, w = 1 s; dotted line: physical equality $\boldsymbol{I} \equiv \{s_1 = s_2\}$; thick curve: subjective equality $\boldsymbol{E}_{1/2}$.

where $U_t \equiv \int_0^t \mathrm{e}^{-\kappa(t-\tau)}\,\mathrm{d}W_\tau$ is the standard Uhlenbeck–Ornstein process. Therefore, terminal states of klepsydrae 1 and 2 at time $t = s_1 + w + s_2$ are independent, normally distributed random variables, $K_k \sim \mathcal{N}(\mu_k, \nu_k^2)$ $(k = 1, 2)$, with

$$\left.\begin{aligned} \mu_1 &= \tfrac{i}{\kappa}\,(1 - \mathrm{e}^{-\kappa s_1})\,\mathrm{e}^{-\kappa(w+s_2)}\,, & \nu_1^2 &= \tfrac{\sigma^2}{2\kappa}\,(1 - \mathrm{e}^{-2\kappa s_1})\,\mathrm{e}^{-2\kappa(w+s_2)}\,, \\ \mu_2 &= \tfrac{i}{\kappa}\,(1 - \mathrm{e}^{-\kappa s_2})\,, & \nu_2^2 &= \tfrac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-2\kappa s_2})\,. \end{aligned}\right\} \tag{2}$$

Psychometric function

Of interest is probability of a certain response, *e. g.*, $J = 1$, as a function of durations s_1 and s_2, that is, the "psychometric function" (PMF),

$$\Psi(s_1, s_2) \equiv \mathsf{P}\{J = 1 \mid s_1, s_2\} \equiv \mathsf{P}\{s_1 \succcurlyeq s_2\}\,.$$

Since K_1 and K_2 are independent, their difference $K_1 - K_2 \sim \mathcal{N}(\mu_1 - \mu_2, \nu_1^2 + \nu_2^2)$, and thus

$$\Psi(s_1, s_2) = \mathsf{P}\{K_1 - K_2 > 0\} = \Phi(\zeta)\,,$$

where $\Phi(\cdot)$ is the Gaussian c. d. f., and its argument[1]

$$\zeta = \frac{\mu_1 - \mu_2}{\sqrt{\nu_1^2 + \nu_2^2}} \tag{3}$$

is a function of durations s_1, s_2, depending on parameters κ, i, σ, and the inter-stimulus interval w. Inserting expressions for $\mu_1, \mu_2, \nu_1, \nu_2$ from (2) into (3) and writing, for brevity,[2]

$$\mathfrak{C} \equiv \gamma\sqrt{\frac{2}{\kappa}}\,, \tag{4}$$

gives

$$\zeta = \mathfrak{C}\,\frac{(1 - \mathrm{e}^{-\kappa s_1})\,\mathrm{e}^{-\kappa w} - (\mathrm{e}^{\kappa s_2} - 1)}{\sqrt{(1 - \mathrm{e}^{-2\kappa s_1})\,\mathrm{e}^{-2\kappa w} + (\mathrm{e}^{-2\kappa s_2} - 1)}}\,.$$

In the limiting case $\kappa \to 0$, eq. (3) attains the form

$$\zeta = \gamma\,\frac{s_1 - s_2}{\sqrt{s_1 + s_2}}\,,$$

in which case the PMF is exactly antisymmetric, *i. e.*, $\Psi(s_1, s_2) + \Psi(s_2, s_1) = 1$.

Relation to duration reproduction

The psychometric function can be visualised as a bunch of curves $\boldsymbol{E}_p \equiv \{(s_1, s_2) \mid \Psi(s_1, s_2) = p\}$ with $p \in (0, 1)$ (Fig. 1c). Of special importance is the 'manifold of subjective equality' $\boldsymbol{E}_{1/2}$, where $\mathsf{P}\{s_1 \gtrdot s_2\} = \mathsf{P}\{s_1 \lessdot s_2\}$, therefore $\zeta = 0$, and thus

$$(1 - \mathrm{e}^{-\kappa s_1})\,\mathrm{e}^{-\kappa w} = \mathrm{e}^{\kappa s_2} - 1\,.$$

This is equivalent to $s_2 = \mathrm{krf}(s_1, w)$, where $\mathrm{krf}(\cdot, \cdot)$ is the 'klepsydraic reproduction function,' predicting mean response times in the model of duration reproduction [2].[3] While in our previous work [5] the equivalence of reproduction and pairwise comparison was asserted on merely intuitive grounds, here the equivalence of the two experimental paradigms is rigorously proven.

Estimation of model parameters

Suppose we are given data from a duration discrimination experiment with duration pairs (s_1, s_2) drawn from a one-dimensional stimulus manifold $\boldsymbol{S}$. Estimates of parameters κ, γ can be obtained via the maximum-likelihood method with the exact PMF $\Psi(s_1, s_2) = \Phi(\zeta(s_1, s_2))$ of the SDKM, using a special iterative fitting procedure. This, however, can be avoided by a simple linearisation strategy sketched in the following.

For definiteness, suppose that $\boldsymbol{S}$ is given by a smooth, parametrised curve $x \mapsto \boldsymbol{s}(x) = (s_1(x), s_2(x))$. Close to any $x = \xi$, the function $x \mapsto \zeta(\boldsymbol{s}(x))$ is well approximated by its tangent $x \mapsto \zeta(\boldsymbol{s}(\xi)) + \beta\,(x - \xi)$, where the slope β is given by the inner product of the gradient of ζ and the derivative of $\boldsymbol{s}$ at $\boldsymbol{s}(\xi)$ and ξ, respectively, $\beta = \nabla\zeta(\boldsymbol{s}(\xi)) \cdot \boldsymbol{s}'(\xi)$.

Substantial simplification occurs at the intersection of $\boldsymbol{S}$ and $\boldsymbol{E}_{1/2}$, *i. e.*, at the PSE $\boldsymbol{s}^\circ \equiv (s_1^\circ, s_2^\circ) = (s_1(x^\circ), s_2(x^\circ))$. To see this, let us write $\zeta = \gamma\, u/\sqrt{v}$, where for fixed κ and w,

$$\begin{aligned} u(s_1, s_2) &= (1 - \mathrm{e}^{-\kappa s_1})\,\mathrm{e}^{-\kappa w} - (\mathrm{e}^{\kappa s_2} - 1), \\ v(s_1, s_2) &= \frac{\kappa}{2}\,\big((1 - \mathrm{e}^{-2\kappa s_1})\,\mathrm{e}^{-2\kappa w} + \mathrm{e}^{2\kappa s_2} - 1\big)\,. \end{aligned}$$

At the PSE we have $\zeta(s_1^\circ, s_2^\circ) = 0$, hence $u(s_1^\circ, s_2^\circ) = 0$, and thus

$$\nabla\zeta(\boldsymbol{s}^\circ) = \gamma\,\frac{\nabla u}{\sqrt{v}}(\boldsymbol{s}^\circ) - \frac{\gamma}{2}\,\frac{u\nabla v}{v^{3/2}}(\boldsymbol{s}^\circ) = \gamma\,\frac{\nabla u}{\sqrt{v}}(\boldsymbol{s}^\circ),$$

so that at least near the PSE the following linearisation applies,

$$\zeta(\boldsymbol{s}(x)) \approx \gamma\,\rho^\circ\,(x - x^\circ), \quad \text{where } \rho^\circ \equiv \frac{1}{\sqrt{v(\boldsymbol{s}^\circ)}}\nabla u(\boldsymbol{s}^\circ) \cdot \boldsymbol{s}'(x^\circ)\,.$$

Thus, if instead of the exact PMF we fit a standard PMF of the form $\Phi\big(\frac{x-\theta}{\omega}\big)$ to the data, then parameters θ (PSE) and ω^{-1} (discrimination acuity) correspond to x° and $\beta^\circ = \gamma\,\rho^\circ$, respectively, and the fitted values $\widehat{\theta}$, $\widehat{\omega}$ yield estimates $\widehat{\kappa}$, $\widehat{\gamma}$ of the SDKM parameters, as follows. — First, $\widehat{\theta}$ determines our estimate of the PSE, $\widehat{\boldsymbol{s}} \equiv (s_1(\widehat{\theta}), s_2(\widehat{\theta}))$, on the stimulus curve $\boldsymbol{S}$. Then $\widehat{\kappa}$ is obtained as the solution of the equation $u(\widehat{s}_1, \widehat{s}_2) = 0$ for the single quantity that was not yet fixed, that is κ.[4] Second, on substituting the arguments of ρ° by their respective estimates $\widehat{s}_1, \widehat{s}_2, \widehat{\kappa}$, one gets a value $\widehat{\rho}$ free of unknowns,[5] and we set $\widehat{\gamma} = (\widehat{\omega}\,\widehat{\rho})^{-1}$.

Let us summarise some noteworthy features of our estimates. First, the estimate of the relaxation time, $\widehat{\kappa}^{-1}$, depends on the point where $\boldsymbol{S}$ and $\boldsymbol{E}_{1/2}$ intersect (PSE), but not otherwise on the stimulus manifold. On the contrary, the estimated signal-to-noise ratio (or subject's discrimination acuity) $\widehat{\gamma}$ *does* depend on the choice of $\boldsymbol{S}$, via the inner product appearing in the numerator of ρ° ($\widehat{\rho}$) which involves the derivative of the stimulus curve. Actually, the dependence is purely

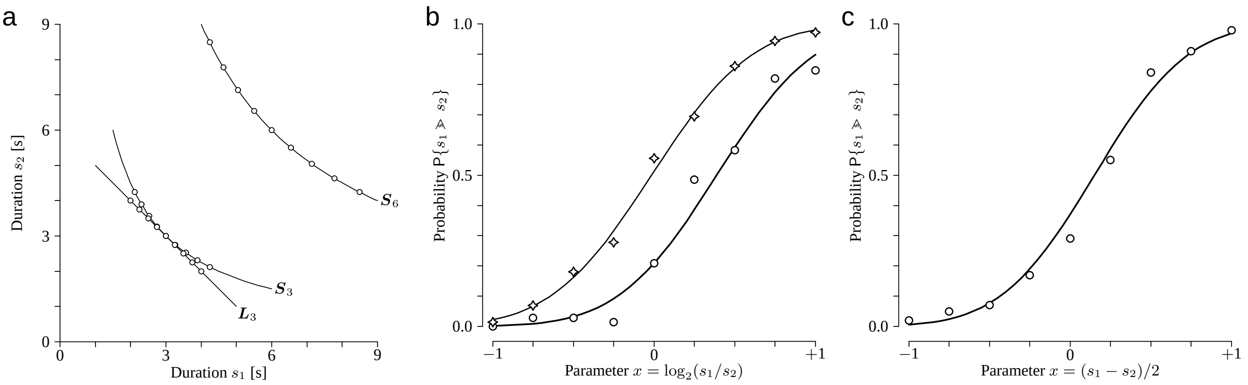

Figure 2. Experimental data on duration discrimination. (a) Stimuli employed in the two reported studies. (b) PMFs form the first study; thin curve, $\diamond$: short durations ($\boldsymbol{S}_3$), thick curve, $\circ$: long durations ($\boldsymbol{S}_6$); each data point based on 72 trials (9 subjects $\times$ 8 repetitions). (c) PMF from the follow-up study; each data point based on 100 trials (10 subjects $\times$ 10 repetitions).

geometrical, involving only the *angle* at which the two manifolds intersect; this follows from the invariance of $\widehat{\gamma}$ under reparametrisations of the stimulus curve. The cosine of this angle, and hence, for any fixed γ, the discrimination acuity ω^{-1} is maximised if $\boldsymbol{S}$ is orthogonal to $\boldsymbol{E}_{1/2}$ at the PSE. We observe, in passing, that this confirms the golden rule of psychophysical experimentation: the experimental design should be derived from a model of the phenomenon under consideration. If this rule is not always followed in experimental studies, it is mostly due to the lack of a theoretical model.

In standard designs of sensory discrimination experiments, one of the paired stimuli is usually held constant (‘standard’) while the other stimulus is varied. Thus there are two experimental designs in which either (I) the variable $x \equiv s_1$ precedes the standard $s_2 = s =$ const, or (II) the standard $s_1 = s =$ const precedes the variable $x \equiv s_2$. Accordingly, we obtain two partial PMFs,

$$\dot{\Psi}_{\mathrm{I}}(x) = \Psi(x, s), \quad \dot{\Psi}_{\mathrm{II}}(x) = \Psi(s, x). \tag{5}$$

In these two cases, $\boldsymbol{S}$ is a straight segment parallel to one of the two axes in the plane of duration pairs (s_1, s_2). There is, however, no particular reason for these special experimental designs except for tradition and ease of technical realisation.

Two data-based examples

The procedures described above are illustrated by results of two experimental studies. In the first study [5], pairs of stimulus durations were chosen from two manifolds $\boldsymbol{S}_c \equiv \{(s_1, s_2) \mid s_1\, s_2 = c^2\}$, with c = 3 and 6 s, resp., parametrised by logarithmic ratio $x = \log_2(s_1/s_2)$. In a follow-up study, stimuli were taken from a manifold $\boldsymbol{L}_c \equiv \{(s_1, s_2) \mid s_1 + s_2 = 2\,c\}$, with $c = 3$ s, parametrised by difference $x = (s_1 - s_2)/2$ (Fig. 2a). In both studies, the stimulus parameter x was varied at nine levels, displaced symmetrically w. r. t. 0. Empirical PMFs fitted to group-averaged responses are shown in Fig. 2b, c,[6] and the respective estimates are summarised in Table 1 on the next page.

Except of the data subset $\boldsymbol{S}_3$ in the 1st study, where the PSE is almost exactly located at $\boldsymbol{I}$, we see a pronounced asymmetry of the PMF in two of the three data sets. In these two cases, the estimates $\widehat{\kappa}$ are in the same order of magnitude as those based on duration reproduction data [2]; accordingly, the relaxation times range from $\sim$30 to 45 s. In addition, estimates of the signal-to-noise ratio $\widehat{\gamma}$ are in good agreement within the three data sets, with the respective diffusion times being in the range $\sim$140 to 320 ms.

Table 1.

Study no. (stimulus)	Subjective equality $\widehat{s}_1$ [s]	$\widehat{s}_2$ [s]	$\widehat{s}_1/\widehat{s}_2$	SDKM parameters $\widehat{\kappa}$ [s^{-1}]	$\widehat{\gamma}$ [s$^{-1/2}$]	Char. times $\widehat{\kappa}^{-1}$ [s]	$\widehat{\gamma}^{-2}$ [s]	Ratio $\mathfrak{C}$
1 ($\boldsymbol{S}_3$)	2.9831	3.0170	0.9888	0	2.4056	∞	0.1728	n. a.
1 ($\boldsymbol{S}_6$)	6.8629	5.2456	1.3083	0.0335	1.7647	29.87	0.3211	40.54
2 ($\boldsymbol{L}_3$)	3.1497	2.8503	1.1050	0.0222	2.6800	45.03	0.1392	27.14

Miscellaneous aspects

Presentation order error

As seen above, the PMF predicted by the SDKM is exactly anti-symmetric only for $\kappa = 0$; otherwise, the PMF is asymmetric and the manifold of subjective equality, $\boldsymbol{E}_{1/2}$, deviates from physical equality, $\boldsymbol{I}$ (Fig. 1c). Therefore, the partial PMFs $\dot{\Psi}_{\mathrm{I}}$ and $\dot{\Psi}_{\mathrm{II}}$ [see eq. (5)] are generally *not identical*; in particular, the PSEs for the partial PMFs are different, $\theta_{\mathrm{I}} > s > \theta_{\mathrm{II}}$, where s is the 'standard' duration. The model thus predicts a *presentation order effect* (POE) (also called 'time order error' or *Zeitfehler* [6]), a well-known phenomenon in time perception as well as in other sensory modalities [7, 8]. Importantly, the POE arises as a natural feature of the model, *all other things being constant*: there is no need for auxiliary assumptions such as response bias, unequal attention allocation, or unequal weighting of the 1st and 2nd stimulus [9].[7]

While the notion of the POE refers to the difference between two special partial PMFs, a more natural measure of the discrimination asymmetry is given by the deviation of the PSE from physical equality (expressed *e. g.* by the ratio $\widehat{s}_1/\widehat{s}_2$), or, in our model, by the value $\widehat{\kappa}$.

Estimates of terminal states

Terminal states accumulated at klepsydræ 1 and 2 may be of interest, especially in studies aiming at neural mechanisms of internal duration representation. The klepsydræ *per se* are unobservable; however, we can set, by convention, $i \equiv 1$, and calculate conditional means of the terminal states, given the durations s_1, s_2, and the subject's response J:[8]

$$\langle K_1 \rangle_{J=j} = \mu_1 - m_j \, \frac{\nu_1^2}{\sqrt{\nu_1^2 + \nu_2^2}} \,, \quad \langle K_2 \rangle_{J=j} = \mu_2 + m_j \, \frac{\nu_2^2}{\sqrt{\nu_1^2 + \nu_2^2}} \,, \tag{6}$$

Here, μ_j, ν_j are specified by eq. (2), and the multipliers m_j are defined as follows: $m_1(\zeta) \equiv -\Phi'(\zeta)/\Phi(\zeta)$, $m_2(\zeta) \equiv \Phi'(\zeta)/(1 - \Phi(\zeta))$, where $\Phi(\cdot)$ is the Gaussian c. d. f.

Quantities defined by eqs. (6) can be correlated, on a trial-by-trial basis, with neurophysiological measurements as, for example, activation of brain areas assessed by fMRI (*cf.* ref. [10]). If the external correlate is expected to reflect inflow intensities rather than accumulated states, it may be reasonable to normalise $\langle K_k \rangle_j$ by the respective durations s_k.

Retrospective judgment of temporal order

Consider a cognitive system for which the temporal order of events is indeterminate unless their order of occurrence becomes biologically or psychologically significant, and only then is evaluated on the basis of elapsed durations. Formally, for two events, E_1 and E_2, occurring at times t_1 and t_2, and being recalled at time t_0, $t_0 - t_1 \gtrdot t_0 - t_2 \Rightarrow E_1 \prec E_2$. While in the model presented above the internal representations of durations were accumulated sequentially, here the model is modified to represent two *parallel* durations. In this latter case the PMF is always symmetrical, even for $\kappa > 0$. Of interest is then discrimination acuity, *i. e.*, the ability to distinguish the succession of events occurred at times $t_{1,2} = \bar{t} \pm \frac{1}{2}\triangle t$. Analysis shows that, for $\kappa = 0$, discrimination acuity is monotonically increasing, proportionally to $\sqrt{\bar{t}}$; whereas in the realistic case $\kappa > 0$, discrimination acuity reaches a maximum for some optimal 'past depth' and then decreases with $\bar{t} \to \infty$. Expectedly, the dimensionless ratio $\mathfrak{C}$ defined by eq. (4) plays the rôle of a critical parameter.

This observation illustrates the interplay between the two characteristic time-scales (diffusion *vs.* relaxation), and possibly opens a psychophysical perspective on the 'inner and outer horizons' of subjective experience of time [11]. A detailed treatment of this topic is reserved for a separate communication.

Notes

[1] In signal detection theory usually represented by symbol d' [7].
[2] So that $\frac{1}{2}\,\mathfrak{C}^2$ is a ratio of the relaxation and diffusion times.
[3] Strictly speaking, this is the 'normal case' of the krf, where parameter $\eta \equiv i_1/i_2 = 1$; but the equivalence between reproduction and comparison demonstrably holds for $\eta \neq 1$ as well.
[4] This equation always has the trivial solution $\kappa = 0$. Under the condition $w \leq \widehat{s}_2$, which can be guaranteed in advance, the equation has a positive solution if and only if $\widehat{s}_1 > \widehat{s}_2$. In that case the positive solution is uniquely determined and taken as our estimate, $\widehat{\kappa}$. If there is no positive solution we set $\widehat{\kappa} = 0$.
[5] If $\widehat{\kappa} = 0$, we set $\widehat{\rho} = (s_1'(\widehat{\theta}) - s_2'(\widehat{\theta}))/(s_1(\widehat{\theta}) + s_2(\widehat{\theta}))^{1/2}$, as suggested by l'Hôpital's rule.
[6] In the original communication [5] the PMFs were defined as $\mathsf{P}\{s_1 \lessdot s_2\}$, and plotted against $-x$; as a consequence, the curves shown there are inverted images of the curves in Fig. 2b.
[7] Hellström in [9] argued that "comparison is not just subtraction." In our model, the comparison *is* just a subtraction [*cf.* the nominator in eq. (3)], but the operands are non-linear functions of physical durations.
[8] We skip details of derivation of formulæ (6) because of space limitations.

References

[1] J. Wackermann, W. Ehm, & J. Späti, 'The "klepsydra model" of internal time representation,' in *Fechner Day 2003*, ed. by B. Berglund & E. Borg, Stockholm: International Society for Psychophysics, 2003, pp. 331–336.

[2] J. Wackermann & W. Ehm, 'The dual klepsydra model of internal time representation and time reproduction,' *Journal of Theoretical Biology* **239**, 482–493, 2006.

[3] W. Ehm & J. Wackermann, 'Analysis of a stochastic version of the "klepsydra model",' in *Fechner Day 2003*, ed. by B. Berglund & E. Borg, Stockholm: International Society for Psychophysics, 2003, pp. 59–64.

[4] W. Ehm & J. Wackermann, 'Estimating the parameters of the stochastic klepsydra model,' in *Fechner Day 2004*, ed. by A. M. Oliveira *et al.*, Coimbra: International Society for Psychophysics, 2004, pp. 350–355.

[5] J. Wackermann & J. Späti, 'Asymmetry of the discrimination function for temporal durations in human subjects,' *Acta Neurobiologiae Experimentalis* **66**, 245–254, 2006.

[6] G. Th. Fechner, *Elements of Psychophysics*, Vol. I (transl. by H. E. Adler) New York: Holt, 1966.

[7] G. A. Gescheider, *Psychophysics: The Fundamentals*, 3rd ed., Mahwah NJ: Lawrence Erlbaum, 1997.

[8] Å. Hellström, 'The time-order error and its relatives: Mirrors of cognitive processes in comparing,' *Psychological Bulletin* **97**, 35–61, 1985.

[9] Å. Hellström, 'Comparison is not just subtraction: Effects of time- and space-order on subjective stimulus difference,' *Perception & Psychophysics* **65**, 1161–1177, 2003.

[10] R. Jech, P. Dušek, J. Wackermann, & J. Vymazal, 'Cumulative blood oxygenation level-dependent signal changes support the "time accumulator" hypothesis,' *NeuroReport* **16**, 1467–1471, 2005.

[11] J. Wackermann, 'Inner and outer horizons of time experience,' *Spanish Journal of Psychology* **10**, 20–32, 2007.